Math 412, Fall 2023 – Homework 9

Due: Wednesday, November 29th, at 9:00AM via Gradescope

Instructions: Students taking the course for three credit hours (undergraduates, most graduate students) should choose four of the following five problems to solve and turn in–if you do all five, only the first four will be graded. Graduate students taking the course for four credits should solve all five. Problems that use the word "describe", "determine", "show", or "prove" require proof for all claims.

1. For a chess piece Q, *the Q-graph* is the graph whose vertices are the squares of the chess board and the two squares are adjacent if Q can move from one of them to the other in one move. Find the chromatic number of the Q-graph when Q is (a) the king, (b) a rook, (c) a bishop, (d) a knight.

2. Prove or disprove: for every n-vertex graph G, $\chi(G) \leq \omega(G) + \frac{n}{\alpha(G)}$.

3. Let G be a simple graph. Prove that the chromatic polynomial $p(k) := \chi(G; k)$ of G satisfies $p'(0) \neq 0$ if and only if G is connected. (*Hint: Use Theorem 5.3.8 and its proof. Note that that result doesn't guarantee that coefficients are nonzero*).

4. Let G be a plane graph, and let G^* be its dual graph. Prove the following:

   (a) G^* is connected.

   (b) If G is connected, then each face of G^* contains exactly one vertex of G. (*Hint: Use Euler's Formula*)

   (c) $(G^*)^* \cong G$ if and only if G is connected.

5. Let P be a polyhedron such that

   - The faces of P are all either pentagons or hexagons.
   - Each vertex of P has degree 3, and lies on the boundary of one pentagon and two hexagons.

   Determine how many vertices, edges, and faces P has, and how many of these faces are pentagonal vs. hexagonal.