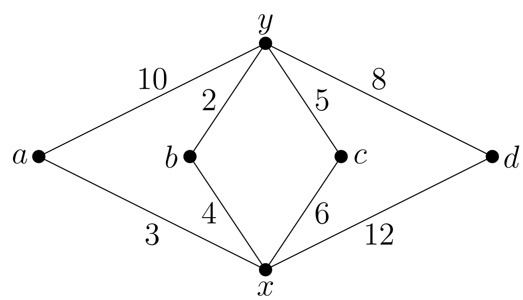
Math 412, Fall 2023 – Homework 6

Due: Wednesday, October 11th, at 9:00AM via Gradescope

Instructions: Students taking the course for three credit hours (undergraduates, most graduate students) should choose four of the following five problems to solve and turn in–if you do all five, only the first four will be graded. Graduate students taking the course for four credits should solve all five. Problems that use the word "describe", "determine", "show", or "prove" require proof for all claims.

1. Consider the following graph.



   (a) Use Kruskal's algorithm to construct a minimal spanning tree, and find its weight. Show your work step by step: which edge is considered at each step of the algorithm, is it accepted or rejected, and why?

   (b) Use Dijkstra's algorithm to find the distance from y to every vertex. Again, show your work step by step, including the set S and the values of the function t at each step.

2. For any spanning tree T in a weighted graph G, let

$$m(T) = \max_{e \in E(T)} \mathrm{wt}(e).$$

   Further, let

$$x(G) = \min_{T} m(T),$$

   where the minimum is over all spanning trees of G.

   (a) If T is a minimal spanning tree of G, prove that $m(T) = x(G)$.

(b) Give an example of a graph G and a *non-minimal* spanning tree T such that $m(T) = x(G)$.

3. Let G be an X,Y-bigraph, i.e. a bipartite graph whose partite sets are X and Y. If $|X| = |Y|$, prove that there exists a subset $S \subseteq X$ with $|N(S)| < |S|$ if and only if there exists a subset $T \subseteq Y$ with $|N(T)| < |T|$.

4. Let G be a graph. Prove that the number of edges in every maximal matching in G is at least half the number of edges of a maximum matching of G.

5. Let D be a digraph. Prove that there exist pairwise disjoint cycles in D such that each vertex of D lies in exactly one of the cycles if and only if

$$|N^+(S)| \geq |S| \text{ for all } S \subseteq V(D).$$