Math 412, Fall 2023 – Homework 5

Due: Wednesday, October 4th, at 9:00AM via Gradescope

Instructions: Students taking the course for three credit hours (undergraduates, most graduate students) should complete **all three** of the following problems. Graduate students taking the course for four credits should contact the instructor. The first problem will count double in the grading. Problems that use the word "describe", "determine", "show", or "prove" require proof for all claims.

1. In this problem, we will prove the weighted Matrix Tree Theorem by using the unweighted analogue. Recall the definitions from class of the tree sum $\tau(G)$ and the Laplacian $L^i(G)$ for weighted graphs.

   (a) Let $f(x_1, \ldots, x_k) : \mathbb{R}^k \to \mathbb{R}$ and $g(x_1, \ldots, x_k) : \mathbb{R}^k \to \mathbb{R}$ be polynomials in k-variables $x_1, \ldots, x_k$. If for all k-tuples $(a_1, \ldots, a_k)$ of positive integers, we have $f(a_1, \ldots, a_k) = g(a_1, \ldots, a_k)$, prove that f and g are equal for *any* inputs. *[Hint: think of f and g as polynomials only in x_k, and consider $f - g$. Use induction on k.]*

   (b) Let G be a weighted loopless graph with positive integer weights. Using the (unweighted) Matrix Tree Theorem, prove that for any i, $\tau(G) = \det L^i(G)$. *[Hint: consider the unweighted graph G' formed from G by taking each edge e of G with weight w and, and letting G' have w edges with the same endpoints as e.]*

   (c) Combine (a) and (b) to prove the weighted Matrix Tree Theorem: if G is a weighted loopless graph with any weights, then for any i, $\tau(G) = \det L^i(G)$. *[Hint: think of the edge weights as variables, and explain why part (a) holds.]*

2. Let D be a weakly-connected weighted loopless digraph with underlying graph G.

   For any walk W in G, define the signed weight of W to be

$$\operatorname{sgwt}(W) = \sum_{e \in E(W)} \pm \operatorname{wt}(e),$$

   where the sign on $\operatorname{wt}(e)$ is positive if and only if we traverse e from tail to head.

   Fix a vertex $v \in V(G)$. Prove that the following conditions are equivalent.

(K2) For any closed walk C in G, $\operatorname{sgwt}(C) = 0$.

(K2') There exists a unique function $U : V(G) \to \mathbb{R}$ such that $U(v) = 0$ and for all $e \in E(D)$,
$$\mathrm{wt}(e) = U(\text{head of } e) - U(\text{tail of } e).$$

[Hint: if $U(v) = 0$ and W is a walk from v to u, what must be true about $U(u)$?]

3. For the following (unweighted) graph G, compute $\tau(G)$, $L(G)$, $\det L^1(G)$ and $\det L^2(G)$, and thus confirm the Matrix Tree Theorem for this example.