Math 412, Fall 2023 – Homework 4

Due: Wednesday, September 27th, at 9:00AM via Gradescope

Instructions: Students taking the course for three credit hours (undergraduates, most graduate students) should choose four of the following five problems to solve and turn in–if you do all five, only the first four will be graded. Graduate students taking the course for four credits should solve all five. Problems that use the word "describe", "determine", "show", or "prove" require proof for all claims.

1. Let G be a digraph with vertices x and y such that $d^+(x) - d^-(x) = d^-(y) - d^+(y) = 4$, and for all other vertices v, $d^+(v) = d^-(v)$. Prove that there exist paths P_1, P_2, P_3, P_4 from x to y in G, such that none of these paths share an edge.

2. Let T be a tournament having no vertex with indegree 0. Prove that T has at least three kings. *[Hint: Prove that if x is a king in T, then T has another king in $N^-(x)$, the in-neighborhood of x.]*

3. Prove that for each simple graph G the following are equivalent.

   (a) G is a forest (i.e. a simple graph with no cycles).

   (b) Every induced subgraph of G has a vertex of degree at most 1.

   (c) The intersection of any two intersecting paths in G is either empty or a path.

   (d) The number of connected components of G is the number of vertices minus the number of edges.

4. For $n \geq 3$, prove that if an n-vertex graph G has three vertices v_1, v_2, v_3 such that the subgraphs $G \setminus v_1, G \setminus v_2$ and $G \setminus v_3$ are acyclic, then G has at most one cycle.

5. Let T be a tree, and let $B \subseteq V(T)$ be the set of vertices of T with degree at least 3. Prove that the number of leaves ℓ of T equals

$$2 + \sum_{v \in B} (d(v) - 2).$$