Math 412, Fall 2023 – Homework 3

Due: Wednesday, September 13th, at 9:00AM via Gradescope

Instructions: Students taking the course for three credit hours (undergraduates, most graduate students) should choose four of the following five problems to solve and turn in–if you do all five, only the first four will be graded. Graduate students taking the course for four credits should solve all five. Problems that use the word "describe", "determine", "show", or "prove" require proof for all claims.

1. Let G be an n-vertex simple graph, with $n \geq 2$. Determine the maximum possible number of edges in G for each of the following conditions:

   (a) G has an independent set of size a.

   (b) G has exactly k connected components.

   (c) G is disconnected.

2. Determine for which values of n there exists an n-vertex 5-regular simple connected graph.

3. Consider the n-dimensional hypercube Q_n. Let C be a cycle of length $2r$ in Q_n for some $r \leq n$. Prove that C is contained in an r-dimensional hypercube $Q_r \subseteq Q_n$.

4. Using the type of argument that we used to prove Mantel's Theorem in Friday's lecture (see posted lecture notes), prove that for $n \geq 1$ the only n-vertex triangle free simple graph with the maximum possible number of edges is $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$.

5. Using Problem 4, determine the minimum number of edges for any n-vertex connected graph with no independent set of size 3 or larger.